

# A geometric approach to tau-functions of difference Painlevé equations

Teruhisa TSUDA

Faculty of Mathematics, Kyushu University,  
Hakozaki, Fukuoka 812-8581, Japan.  
tudateru@math.kyushu-u.ac.jp

April 7, 2008

## Abstract

We present a unified description of birational representation of Weyl groups associated with T-shaped Dynkin diagrams, by using a particular configuration of points in the projective plane. A geometric formulation of tau-functions is given in terms of defining polynomials of certain curves. If the Dynkin diagram is of affine type ( $E_6^{(1)}$ ,  $E_7^{(1)}$  or  $E_8^{(1)}$ ), our representation gives rise to the difference Painlevé equations.

## Introduction

It is classically well-known that a certain birational representation of Weyl groups arises from point configurations. Let  $X_{m,n}$  be the configuration space of  $n$  points of the projective space  $\mathbb{P}^{m-1}$  in *general* position. Then, the Weyl group corresponding to the Dynkin diagram  $T_{2,m,n-m}$  (see Figure 1) acts birationally on  $X_{m,n}$  and is generated by permutations of  $n$  points and the standard Cremona transformation with respect to each  $m$  points; see [4, 5, 11].

We shall put our attention to the two-dimensional case, that is,  $m = 3$  case. Only if  $n = 9$ , the affine case occurs and the corresponding diagram reads  $T_{2,3,6} = E_8^{(1)}$ . The lattice part of the affine Weyl group  $W(E_8^{(1)})$  provides an interesting discrete dynamical system, called the *elliptic-difference Painlevé equation* [15, 17]. This  $(m, n) = (3, 9)$  case was explored by Sakai [17] (cf. [16]) in order to clarify the geometric nature of the affine Weyl group symmetry of Painlevé equations; moreover he classified all the degeneration of the nine-points configuration in  $\mathbb{P}^2$ , and as a result he completed the whole list of (second-order) discrete Painlevé equations. In this context the top of all the discrete and continuous Painlevé equations is the elliptic-difference one, from which every other can be obtained through an appropriate limiting procedure. The discrete Painlevé equations are divided into three types: difference,  $q$ -difference or elliptic-difference one, according to their corresponding rational surfaces (obtained by blowing up nine points of  $\mathbb{P}^2$ ); see [17]. On the

---

2000 *Mathematics Subject Classification* 14E05, 33E17, 34M55, 37K20, 37K35, 39A10

*Keywords:* birational transformation, difference Painlevé equation, tau-function, Weyl group.

other hand, even in the two-dimensional case ( $m = 3$ ), by considering certain particular configurations of point sets that are not only nine points, one can enjoy Weyl groups associated to more various Dynkin diagrams [12, 20]; see also [21] in the higher dimensional case.

In this paper, we present a unified description of birational representation of Weyl groups associated with T-shaped Dynkin diagrams (see Figure 1), arising from a particular configuration of points in the projective plane. Our construction, in affine case:  $E_6^{(1)}$ ,  $E_7^{(1)}$  and  $E_8^{(1)}$ , is relevant to the *difference* Painlevé equations; we refer to [20] in the case of *q*-difference ones.

In Section 1, we first start from  $\ell_1 + \ell_2 + \ell_3$  points in  $\mathbb{P}^2$  restricted on three lines  $L_i$  ( $i = 1, 2, 3$ ) meeting at a single point, where the  $\ell_i$  points lie on each  $L_i$ . Let  $X$  be the rational surface obtained from  $\mathbb{P}^2$  by blowing up the  $\ell_1 + \ell_2 + \ell_3$  points. We can find naturally the root lattice of type  $T_{\ell_1, \ell_2, \ell_3}$  included in the Picard group  $\text{Pic}(X)$  of the surface. The corresponding Weyl group  $W = W(T_{\ell_1, \ell_2, \ell_3})$  acts linearly on  $\text{Pic}(X)$ . Next, in order to lift this linear action  $W : \text{Pic}(X) \hookrightarrow$  to the level of birational transformations on the surface  $X$  itself, we introduce a geometric formulation of *tau-functions*. Recall that a smooth rational curve with self-intersection  $-1$  is said to be an *exceptional curve* (of the first kind); see e.g. [2]. An element of  $W$  induces a permutation among exceptional curves on  $X$ , as analogous to the classical subject: 27 lines on a cubic surface and a Weyl group of type  $E_6$ . The tau-function is defined by means of appropriately *normalized* defining functions of the exceptional curves; see (1.3) and (1.4). Today the notion of tau-functions is universal in the field of integrable systems and it is characterized in several directions; however, such an algebro-geometric idea of tau-functions first appeared in the study of the elliptic-difference Painlevé equation by Kajiwara et al. [9] (see also [11, 20, 21] for subsequent development). By imposing on the tau-function a certain compatibility with the linear action  $W : \text{Pic}(X) \hookrightarrow$ , we finally obtain a birational representation of  $W$  acting on the tau-function and the *homogeneous* coordinates of  $\mathbb{P}^2$  (Theorem 1.1). Section 2 concerns the affine case ( $E_6^{(1)} = T_{3,3,3}$ ,  $E_7^{(1)} = T_{4,4,2}$  and  $E_8^{(1)} = T_{6,3,2}$ ); we demonstrate how to derive the difference Painlevé equation for each.

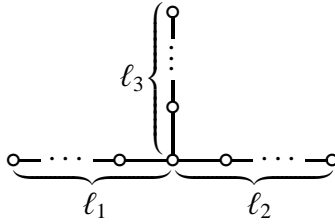


Figure 1: Dynkin diagram  $T_{\ell_1, \ell_2, \ell_3}$

## 1 Birational representation of Weyl groups and tau-functions

We begin with considering the  $\ell_1 + \ell_2 + \ell_3$  points in the complex projective plane  $\mathbb{P}^2$  restricted on three lines  $L_i$  ( $i = 1, 2, 3$ ) meeting at a single point  $P_0$ , where we arrange  $\ell_i$  points on each  $L_i \setminus P_0$ . Let  $\mathbf{x} = [x_1 : x_2 : x_3]$  denote the homogeneous coordinates of  $\mathbb{P}^2$ . By  $\text{PGL}_3(\mathbb{C})$ -action, we can normalize, without loss of generality, the three lines and the point configuration  $\{P_i^m\}_{1 \leq i \leq \ell_m; m=1,2,3}$  to

be as follows:

$$\begin{aligned} L_i &= \{x_j = x_k\} \quad \text{for } \{i, j, k\} = \{1, 2, 3\}, \\ P_i^1 &= [b_i^1 + 1 : b_i^1 : b_i^1] \quad (1 \leq i \leq \ell_1), \\ P_i^2 &= [b_i^2 : b_i^2 + 1 : b_i^2] \quad (1 \leq i \leq \ell_2), \\ P_i^3 &= [b_i^3 : b_i^3 : b_i^3 + 1] \quad (1 \leq i \leq \ell_3), \end{aligned}$$

where  $3b_1^m = -1 + a_0 + c_1 + c_2 + c_3 - 3c_m$  and  $b_{i+1}^m = b_i^m + a_i^m$  ( $1 \leq i \leq \ell_m - 1; m = 1, 2, 3$ ) with  $c_m = \sum_{i=1}^{\ell_m-1} (1 - \frac{i}{\ell_m}) a_i^m$ . Here  $\mathbf{a} = (a_0, (a_i^m)_{1 \leq i \leq \ell_m-1; m=1,2,3}) \in \mathbb{C}^{\ell_1+\ell_2+\ell_3-2}$  play roles of free parameters. We will show later the meaning of  $\mathbf{a}$  as root variables corresponding to the Dynkin diagram  $T_{\ell_1, \ell_2, \ell_3}$ .

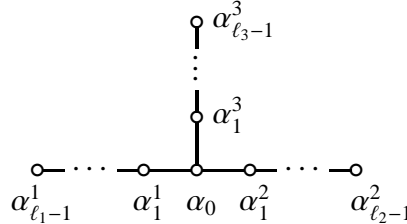
Let  $\pi : X \rightarrow \mathbb{P}^2$  be the blow-up at the  $\ell_1 + \ell_2 + \ell_3$  points  $P_i^m$ . The *Picard group* of the surface  $X$  thus obtained is expressed as

$$\text{Pic}(X) = \mathbb{Z}h \oplus \bigoplus_{\substack{i=1, \dots, \ell_m; \\ m=1, 2, 3}} \mathbb{Z}e_i^m \quad (\cong \mathbb{Z}^{\ell_1+\ell_2+\ell_3+1}).$$

Note that in this case  $\text{Pic}(X)$  is isomorphic to the second cohomology group  $H^2(X, \mathbb{Z})$  because  $X$  is a rational surface. Here we denote by  $h$  the linear equivalent class of  $\pi^{-1}$  of a line and by  $e_i^m$  the class of exceptional curve  $\pi^{-1}(P_i^m)$ . The intersection form  $(|) : \text{Pic}(X) \times \text{Pic}(X) \rightarrow \mathbb{Z}$  is given by

$$(h|h) = 1, \quad (e_i^m | e_j^n) = -\delta_{i,j} \delta_{m,n}, \quad (h | e_i^m) = 0.$$

The anti-canonical class  $-K_X = 3h - \sum_{i,m} e_i^m$  can be decomposed as  $-K_X = D_1 + D_2 + D_3$ , where the classes  $D_m = h - \sum_{i=1}^{\ell_m} e_i^m$  are represented by (the proper transforms of) the three lines  $L_m$  ( $m = 1, 2, 3$ ). Let  $Q$  be the orthogonal complement of  $\{D_1, D_2, D_3\}$  with respect to the intersection form in  $\text{Pic}(X)$ . Then we see that  $Q$  is a root lattice generated by the  $(-2)$ -vectors  $\alpha_{ij}^m = e_i^m - e_j^m$  and  $\alpha_{ijk} = h - e_i^1 - e_j^2 - e_k^3$ . Moreover we can choose a basis  $B = \{\alpha_0 = \alpha_{111}, \alpha_i^m = \alpha_{i,i+1}^m \ (1 \leq i \leq \ell_m - 1, m = 1, 2, 3)\}$  of  $Q$ . The intersection graph (Dynkin diagram) of  $B$  is of type  $T_{\ell_1, \ell_2, \ell_3}$  and looks as follows:



We define the action of the simple reflection corresponding to a root  $\alpha \in Q$  (i.e.,  $\alpha^2 = -2$ ) by

$$R_\alpha(v) = v + (v|\alpha)\alpha$$

for  $v \in \text{Pic}(X)$ . Note that the intersection form is preserved by  $R_\alpha$ . We also prepare the notations  $s_0 = R_{\alpha_0}$  and  $s_i^m = R_{\alpha_i^m}$ , for convenience. We summarize below the linear action of generators  $s_0$  and  $s_i^m$  on the basis of  $\text{Pic}(X)$ :

$$\begin{aligned} s_0(h) &= 2h - e_1^1 - e_1^2 - e_1^3, \\ s_0(e_1^k) &= h - e_1^i - e_1^j \quad \text{for } \{i, j, k\} = \{1, 2, 3\}, \\ s_i^m(e_{\{i,i+1\}}^m) &= e_{\{i+1,i\}}^m. \end{aligned} \tag{1.1}$$

It is easy to check that the Weyl group  $W = \{R_\alpha\}_{\alpha \in Q} = \langle s_0, s_i^m (1 \leq i \leq \ell_m - 1, m = 1, 2, 3) \rangle$  acting on  $\text{Pic}(X)$  indeed satisfies the fundamental relations [7] specified by the Dynkin diagram  $T_{\ell_1, \ell_2, \ell_3}$ .

Now we shall extend the above linear action of  $W : \text{Pic}(X) \rightarrow \text{Pic}(X)$  to the level of birational transformations on the surface  $X$ . We first fix the action on the root variables  $a_0$  and  $a_i^m (1 \leq i \leq \ell_m - 1, m = 1, 2, 3)$  as

$$\begin{aligned} s_0(a_0) &= -a_0, & s_0(a_1^m) &= a_0 + a_1^m, \\ s_i^m(a_i^m) &= -a_i^m, & s_i^m(a_{i\pm 1}^m) &= a_i^m + a_{i\pm 1}^m, \end{aligned} \quad (1.2)$$

where  $a_0^m = a_0$ . We consider a sub-lattice  $M = \bigsqcup_{n=1,2,3} M_n$  of  $\text{Pic}(X)$ , where

$$\begin{aligned} M_n &\stackrel{\text{def}}{=} W \cdot \{e_1^n\} \\ &= \left\{ \Lambda = dh - \sum_{i,j} \mu_j^i e_j^i \in \text{Pic}(X) \mid \begin{array}{l} \Lambda^2 = -1, (\Lambda|D_n) = 1, (\Lambda|D_m) = 0 (m \neq n) \\ d \geq 0; \text{ if } d > 0, \text{ then } \mu_j^i \geq 0 \text{ for } \forall(i, j). \end{array} \right\}. \end{aligned}$$

A divisor class  $\Lambda \in M \subset \text{Pic}(X)$  is represented by an exceptional curve because  $\Lambda^2 = (\Lambda|K_X) = -1$ . Moreover, each class  $\Lambda = dh - \sum_{i,m} \mu_i^m e_i^m \in M$  corresponds to (the proper transform of) a unique plane curve  $C_\Lambda$  passing through  $P_i^m$  with multiplicity  $\mu_i^m$  as its representative. Let  $F_\Lambda = F_\Lambda(\mathbf{a}; \mathbf{x})$  be the *normalized* defining polynomial of  $C_\Lambda$  such that the following key condition is satisfied:

$$F_\Lambda(\mathbf{a}; 1, 1, 1) = 1. \quad (1.3)$$

Notice that  $C_\Lambda$  never passes through the point  $P_0 = [1 : 1 : 1] = L_1 \cap L_2 \cap L_3$  by definition. Let us take for example  $\Lambda = h - e_i^m - e_j^n \in M$ , which is represented by the line passing through  $P_i^m$  and  $P_j^n$ ; thus we have

$$F_{h-e_i^m-e_j^n}(\mathbf{a}; \mathbf{x}) = (1 + b_i^m + b_j^n)x_\ell - b_i^m x_m - b_j^n x_n \quad \text{for } \{\ell, m, n\} = \{1, 2, 3\}.$$

We introduce new variables  $\tau_i^m$  attached to the points  $P_i^m$ , and prepare a field  $L = K(\mathbf{x}; \boldsymbol{\tau})$  of rational functions in indeterminates  $x_m$  and  $\tau_i^m (m = 1, 2, 3; 1 \leq i \leq \ell_m)$  with coefficient field  $K = \mathbb{C}(\mathbf{a})$ . By means of the normalized defining polynomial, we define a function  $\tau : M \rightarrow L = \mathbb{C}(\mathbf{a})(\mathbf{x}; \boldsymbol{\tau})$ , called the *tau-function*, by the formula

$$\tau(\Lambda) \prod_{i,m} \tau(e_i^m)^{\mu_i^m} = F_\Lambda(\mathbf{a}; \mathbf{x}) \quad (1.4)$$

for  $\Lambda = dh - \sum_{i,m} \mu_i^m e_i^m \in M (d > 0)$ , and

$$\tau(e_i^m) = \tau_i^m. \quad (1.5)$$

By imposing the following assumption:

$$w \cdot \tau(\Lambda) = \tau(w \cdot \Lambda), \quad (1.6)$$

one can fix the action of Weyl group  $W$  on  $L = \mathbb{C}(\mathbf{a})(\mathbf{x}; \boldsymbol{\tau})$  in the following manner. For a rational function  $\varphi(\mathbf{a}; \mathbf{x}; \boldsymbol{\tau}) \in L$ , we suppose that an element  $w \in W$  acts as

$$w \cdot \varphi(\mathbf{a}; \mathbf{x}; \boldsymbol{\tau}) = \varphi(\mathbf{a}.w; \mathbf{x}.w; \boldsymbol{\tau}.w),$$

that is,  $w$  acts on the independent variables from the right. We now determine the action of the generators  $s_0$  and  $s_i^m$  from (1.6) as a necessary condition. In view of  $s_0(e_1^k) = h - e_1^i - e_1^j$  ( $\{i, j, k\} = \{1, 2, 3\}$ ), we have

$$s_0(\tau_1^k) = \frac{F_{h-e_1^i-e_1^j}(\mathbf{a}; \mathbf{x})}{\tau_1^i \tau_1^j}.$$

The other  $\tau_i^m$ 's ( $i \neq 1$ ) are kept still by  $s_0$ -action. Applying  $s_0$  to  $F_{h-e_1^i-e_1^j}(\mathbf{a}; \mathbf{x}) = \tau(h - e_1^i - e_1^j) \tau_1^i \tau_1^j$ , we have

$$\begin{aligned} (\text{LHS}) &= s_0(F_{h-e_1^i-e_1^j}(\mathbf{a}; \mathbf{x})) \\ &= F_{h-e_1^i-e_1^j}(s_0(\mathbf{a}); s_0(\mathbf{x})) \\ &= (1 + \tilde{b}_1^i + \tilde{b}_1^j) s_0(x_k) - \tilde{b}_1^i s_0(x_i) - \tilde{b}_1^j s_0(x_j), \\ (\text{RHS}) &= s_0(\tau(h - e_1^i - e_1^j) \tau_1^i \tau_1^j) \\ &= \tau_1^k \tau(h - e_1^i - e_1^j) \tau(h - e_1^i - e_1^j) \\ &= \frac{F_{h-e_1^i-e_1^j}(\mathbf{a}; \mathbf{x}) F_{h-e_1^i-e_1^j}(\mathbf{a}; \mathbf{x})}{\tau_1^1 \tau_1^2 \tau_1^3}, \end{aligned}$$

where  $\{i, j, k\} = \{1, 2, 3\}$  and

$$\tilde{b}_1^i := s_0(b_1^i). \quad (1.7)$$

Solving the above equations with respect to  $s_0(x_i)$ , we obtain

$$s_0(x_i) = \frac{(1 + \tilde{b}_1^i) f_j f_k + \tilde{b}_1^j f_i f_k + \tilde{b}_1^k f_i f_j}{(1 + \tilde{b}_1^1 + \tilde{b}_1^2 + \tilde{b}_1^3) \tau_1^1 \tau_1^2 \tau_1^3},$$

where

$$f_i := F_{h-e_1^j-e_1^k}(\mathbf{a}; \mathbf{x}) = (1 + b_1^j + b_1^k) x_i - b_1^j x_j - b_1^k x_k \quad \text{for } \{i, j, k\} = \{1, 2, 3\}. \quad (1.8)$$

The action of  $s_i^m$  is much simpler and is realized as (just a permutation of  $\tau$ -variables)

$$s_i^m(\tau_{\{i,i+1\}}^m) = \tau_{\{i+1,i\}}^m, \quad s_i^m(\mathbf{x}) = \mathbf{x}.$$

Summarizing above, we have the

**Theorem 1.1.** (I) Define the birational transformations  $s_0$  and  $s_i^m$  by

$$s_0(\tau_1^i) = \frac{f_i}{\tau_1^j \tau_1^k}, \quad (1.9a)$$

$$s_0(x_i) = \frac{(1 + \tilde{b}_1^i) f_j f_k + \tilde{b}_1^j f_i f_k + \tilde{b}_1^k f_i f_j}{(1 + \tilde{b}_1^1 + \tilde{b}_1^2 + \tilde{b}_1^3) \tau_1^1 \tau_1^2 \tau_1^3}, \quad (1.9b)$$

where  $\{i, j, k\} = \{1, 2, 3\}$ , and

$$s_i^m(\tau_{\{i,i+1\}}^m) = \tau_{\{i+1,i\}}^m. \quad (1.9c)$$

Then (1.9) with (1.2) realize the Weyl group  $W = W(T_{\ell_1, \ell_2, \ell_3})$  over the field  $\mathbb{C}(\mathbf{a})(\mathbf{x}; \boldsymbol{\tau})$ .

(II) Moreover (1.4) and (1.6) are consistent.

*Proof.* (I) By direct computation, we can verify that (1.9) indeed satisfy the fundamental relations of  $W$ .

(II) We will show that the formula (1.4) can be recovered inductively by (1.6) and (1.9). First, if  $\Lambda = h - e_1^i - e_1^j \in M$ , then (1.4) follows immediately from (1.9a). Next we assume (1.4) is true for  $\Lambda \in M$ . It is enough to verify for  $\Lambda' = w(\Lambda)$  where  $w \in \{s_0, s_i^m\}$  is a generator of  $W$ . Since the action of  $s_i^m$  is just a permutation of  $\tau$ -variables (see (1.9c)), we will concentrate our attention on the only nontrivial case  $w = s_0$ . Applying  $s_0$  to (1.4), we have

$$\begin{aligned} s_0(F_\Lambda(\mathbf{a}; \mathbf{x})) &= s_0 \left( \tau(\Lambda) \prod_{i,j} (\tau_j^i)^{\mu_j^i} \right) \\ &= \tau(s_0.\Lambda) \prod_{i,j} (\tau_j^i)^{\mu_j^i} \prod_{i=1,2,3} \left( \frac{s_0(\tau_1^i)}{\tau_1^i} \right)^{\mu_1^i} \\ &= \tau(s_0.\Lambda) \prod_{i,j} (\tau_j^i)^{\mu_j^i} \prod_{i=1,2,3} \left( \frac{f_i}{\tau_1^1 \tau_1^2 \tau_1^3} \right)^{\mu_1^i}. \end{aligned}$$

Noticing  $s_0.\Lambda = \Lambda + (\Lambda|\alpha_0)\alpha_0 = \Lambda + (d - \mu_1^1 - \mu_1^2 - \mu_1^3)(h - e_1^1 - e_1^2 - e_1^3)$ , we can verify (1.4) for  $\Lambda' = s_0.\Lambda$  immediately from the lemma below.  $\square$

**Lemma 1.2.** We have

$$s_0(F_\Lambda(\mathbf{a}; \mathbf{x})) = \frac{F_{s_0.\Lambda}(\mathbf{a}; \mathbf{x}) \prod_{i=1,2,3} (f_i)^{\mu_1^i}}{(\tau_1^1 \tau_1^2 \tau_1^3)^d} \quad (1.10)$$

for  $\Lambda = dh - \sum_{i,j} \mu_j^i e_j^i \in M$ .

*Proof.* The multiplicity of the curve  $C_\Lambda = \{F_\Lambda = 0\}$  at  $P_1^i = \{f_j = f_k = 0\}$  ( $\{i, j, k\} = \{1, 2, 3\}$ ) is given by  $\mu_1^i = \text{ord}_{P_1^i}(F_\Lambda)$ . That is,  $F_\Lambda$  can be expressed as a polynomial in  $f_i$  ( $i = 1, 2, 3$ ) of the form

$$F_\Lambda = \sum_{k_1+k_2+k_3=d} A_{k_1,k_2,k_3} (f_1)^{k_1} (f_2)^{k_2} (f_3)^{k_3}, \quad A_{k_1,k_2,k_3} \in \mathbb{C}(\mathbf{a})$$

such that  $A_{k_1,k_2,k_3} = 0$  unless  $0 \leq k_i \leq d - \mu_1^i$  ( $i = 1, 2, 3$ ). By using  $s_0(f_i) = f_j f_k / (\tau_1^1 \tau_1^2 \tau_1^3)$ , we have

$$s_0(F_\Lambda) \frac{(\tau_1^1 \tau_1^2 \tau_1^3)^d}{\prod_{i=1,2,3} (f_i)^{\mu_1^i}} = \sum_{k_1+k_2+k_3=d} A_{k_1,k_2,k_3} (f_1)^{d-\mu_1^1-k_1} (f_2)^{d-\mu_1^2-k_2} (f_3)^{d-\mu_1^3-k_3} =: G(\mathbf{x}).$$

We let  $\tilde{k}_i = d - \mu_1^i - k_i$ ,  $\tilde{d} = 2d - \mu_1^1 - \mu_1^2 - \mu_1^3$  and  $\tilde{A}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} = A_{k_1, k_2, k_3}$ . The polynomial  $G(\mathbf{x})$  can be then written as

$$G(\mathbf{x}) = \sum_{\tilde{k}_1+\tilde{k}_2+\tilde{k}_3=\tilde{d}} \tilde{A}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} (f_1)^{\tilde{k}_1} (f_2)^{\tilde{k}_2} (f_3)^{\tilde{k}_3},$$

where  $\tilde{A}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} = 0$  unless  $0 \leq \tilde{k}_i \leq d - \mu_1^i = \tilde{d} - (d - \mu_1^j - \mu_1^k)$ . Therefore, the curve  $\{G(\mathbf{x}) = 0\}$  represents the divisor class  $s_0.\Lambda$ . Since  $f_i|_{x=(1,1,1)} = 1$ , the normalizing condition (1.3) of  $F_\Lambda$  yields  $\sum A_{k_1, k_2, k_3} = 1$ . Hence  $G(1, 1, 1) = \sum \tilde{A}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} = \sum A_{k_1, k_2, k_3} = 1$ , and so  $G = F_{s_0.\Lambda}$  as desired.  $\square$

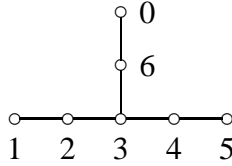
*Remark 1.3.* By virtue of the geometric characterization (1.4) of tau-function, we can trace the resulting value for any iteration of the Weyl group action (Theorem 1.1) by using only the defining polynomials of appropriate plane curves, although it is in general difficult to compute composition of given rational maps. Also, note that the defining polynomial of a curve has a determinantal expression involving information with respect to the multiplicities at enough points; see e.g. [11].

## 2 Affine case and difference Painlevé equations of type E

From Theorem 1.1, one can also realize the Weyl group  $W = W(T_{\ell_1, \ell_2, \ell_3})$  as birational transformations of inhomogeneous coordinates of  $\mathbb{P}^2$ . In this context, tau-functions play roles of *heights* in the sense that the original inhomogeneous coordinate is recovered as a ratio of them. In this section, we consider the affine case:  $E_6^{(1)} = T_{3,3,3}$ ,  $E_7^{(1)} = T_{4,4,2}$  and  $E_8^{(1)} = T_{6,3,2}$ , in which our realization of Weyl groups on inhomogeneous coordinates is equivalent to that of Sakai's formulation [17], and therefore it is relevant to discrete Painlevé equations. Recall that discrete Painlevé equations are divided into three types: difference,  $q$ -difference or elliptic-difference; our setting corresponds to the difference one. For each case, we will derive the *difference Painlevé equations* in terms of appropriately chosen coordinates.

### 2.1 $E_6^{(1)}$ case ( $\ell_1 = \ell_2 = \ell_3 = 3$ )

The Dynkin diagram of type  $E_6^{(1)} = T_{3,3,3}$  is



Let  $(a_0, a_1, a_2, a_3, a_4, a_5, a_6)$  be the root variables of  $E_6^{(1)}$ . The transformation from the original variables  $a_0, a_i^m$  (used in Section 1) is given as follows:

$$a_0 := a_3, \quad a_{\{1,2\}}^1 := a_{\{2,1\}}, \quad a_{\{1,2\}}^2 := a_{\{4,5\}}, \quad a_{\{1,2\}}^3 := a_{\{6,0\}}.$$

We define the action of simple reflections  $s_i$  on the root variables by  $s_i(a_j) = a_j - a_i C_{ij}$ , where  $C_{ij}$  is the Cartan matrix of  $E_6^{(1)}$ . Let  $\iota_i$  ( $i = 1, 2, 3$ ) be the diagram automorphisms defined by  $\iota_1(a_{\{0,1,2,3,4,5,6\}}) = a_{\{5,1,2,3,6,0,4\}}$ ,  $\iota_2(a_{\{0,1,2,3,4,5,6\}}) = a_{\{1,0,6,3,4,5,2\}}$  and  $\iota_3(a_{\{0,1,2,3,4,5,6\}}) = a_{\{0,5,4,3,2,1,6\}}$ .

Let us consider the change of the variables  $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  given by

$$\varphi : [x_1 : x_2 : x_3] \mapsto (u, v) = \left( \frac{F_{h-e_1^1-e_1^2}(\mathbf{a}; \mathbf{x})}{x_2 - x_3}, \frac{F_{h-e_1^1-e_1^2}(\mathbf{a}; \mathbf{x})}{x_1 - x_3} \right).$$

This transformation is obtained by blowing up the points  $P_1^1$  and  $P_1^2$ , and blowing down (the proper transform of) the line  $\{F_{h-e_1^1-e_1^2} = 0\}$  which passes through the two points. Note that the sum of three lines  $L_1 \cup L_2 \cup L_3$ , which is invariant under the action of the Weyl group, is generically transformed into  $\{u = \infty\} \cup \{v = \infty\} \cup \{u = v\}$  by  $\varphi$ .

From Theorem 1.1, we have the following realization of  $\widetilde{W}(E_6^{(1)}) = \langle s_0, \dots, s_6, \iota_1, \iota_2, \iota_3 \rangle$  on  $(u, v)$ :

$$\begin{aligned} s_2(u) &= \frac{u(v - a_2)}{v}, & s_2(v) &= v - a_2, \\ s_3(u) &= u + a_3, & s_3(v) &= v + a_3, \\ s_4(u) &= u - a_4, & s_4(v) &= \frac{v(u - a_4)}{u}, \\ \iota_1(u) &= -u - a_3, & \iota_1(v) &= \frac{v(u + a_3)}{u - v}, \\ \iota_2(u) &= \frac{u(v + a_3)}{v - u}, & \iota_2(v) &= -v - a_3, \\ \iota_3(u) &= v, & \iota_3(v) &= u. \end{aligned}$$

Here the symbol  $\widetilde{W}$  stands for the extended group of  $W$  by its diagram automorphisms. The birational action of a translation in  $\widetilde{W}(E_6^{(1)})$  yields the difference Painlevé equation. We take an element

$$T = (\iota_3 \iota_1 s_6 s_0 s_3 s_4 s_6 s_3 s_5 s_4)^2$$

acting on the root variables as their shifts:  $T(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = (a_0, a_1, a_2 + \delta, a_3, a_4 - \delta, a_5, a_6)$ , where  $\delta = a_0 + a_1 + 2a_2 + 3a_3 + 2a_4 + a_5 + 2a_6$  corresponds to the null root. Let  $\bar{u} = T(u)$  and  $\underline{v} = T^{-1}(v)$ . Then we have the system of functional equations:

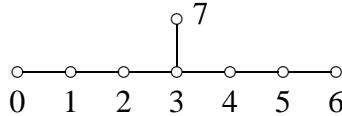
$$(\bar{u} - v)(u - v) = \frac{v(v + a_3)(v + a_3 + a_6)(v + a_3 + a_6 + a_0)}{(v - a_2)(v - a_1 - a_2)}, \quad (2.1a)$$

$$(\underline{v} - u)(v - u) = \frac{u(u + a_3)(u + a_3 + a_6)(u + a_3 + a_6 + a_0)}{(u - a_4)(u - a_4 - a_5)}. \quad (2.1b)$$

This is called the *difference Painlevé equation of type  $E_6^{(1)}$* .

## 2.2 $E_7^{(1)}$ case ( $\ell_1 = \ell_2 = 4, \ell_3 = 2$ )

The Dynkin diagram of type  $E_7^{(1)} = T_{4,4,2}$  is



Let  $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$  be the root variables of  $E_7^{(1)}$ . Here we have replaced the variables  $a_0, a_i^m$  (used in Section 1) as

$$a_0 := a_3, \quad a_{\{1,2,3\}}^1 := a_{\{2,1,0\}}, \quad a_{\{1,2,3\}}^2 := a_{\{4,5,6\}}, \quad a_1^3 := a_7.$$

Define the action of simple reflections  $s_i$  on the root variables by  $s_i(a_j) = a_j - a_i C_{ij}$ , where  $C_{ij}$  is the Cartan matrix of  $E_7^{(1)}$ . Let  $\iota$  be the diagram automorphism defined by  $\iota(a_{\{0,1,2,3,4,5,6,7\}}) = a_{\{6,5,4,3,2,1,0,7\}}$ .



We first blow up the two points  $P_1^3, P_2^3$  and blow down (the proper transform of) the line  $L_3 = \{x_1 = x_2\}$ . This procedure yields the birational map  $\varphi_1 : \mathbb{P}^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  given by

$$\varphi_1 : [x_1 : x_2 : x_3] \mapsto (u_1, v_1) = \left( \frac{(b_1^3 + 1)(x_1 + x_2) - 2b_1^3 x_3}{x_1 - x_2}, \frac{(b_2^3 + 1)(x_1 + x_2) - 2b_2^3 x_3}{x_1 - x_2} \right).$$

In addition, we apply

$$\varphi_2 : (u_1, v_1) \mapsto (u, v) = \left( u_1 + \frac{a_0 + 2a_1 + 3a_2 - 3a_4 - 2a_5 - a_6}{4}, v_1 + \frac{a_0 + 2a_1 + 3a_2 - 3a_4 - 2a_5 - a_6}{4} \right).$$

Note that  $L_1 \cup L_2 \cup L_3$  is generically transformed by  $\varphi = \varphi_2 \circ \varphi_1$  into two curves  $\{u - v + a_7 = 0\} \cup \{u - v - a_7 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree  $(1, 1)$  tangent to each other at  $(\infty, \infty)$ .

From Theorem 1.1, we have the following realization of  $\widetilde{W}(E_7^{(1)})$  on  $(u, v)$ :

$$\begin{aligned} s_2(u) &= u - a_2, & s_2(v) &= v - a_2, & s_3(v) &= \tilde{v}, & s_4(u) &= u + a_4, & s_4(v) &= v + a_4, \\ \iota(u) &= -u, & \iota(v) &= -v, \end{aligned}$$

where  $\tilde{v} = \tilde{v}(u, v)$  is a rational function determined by

$$\frac{u - \tilde{v} + a_3 + a_7}{u - \tilde{v} - a_3 - a_7} = \frac{(u + a_3)(u - v + a_7)}{(u - a_3)(u - v - a_7)}.$$

Let us take an element

$$T = (s_7 s_3 s_2 s_1 s_0 s_4 s_5 s_6 s_3 s_2 s_1 s_4 s_5 s_3 s_2 s_4 s_3)^2 \in W(E_7^{(1)})$$

acting on the root variables as  $T(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (a_0, a_1, a_2, a_3 - \delta, a_4, a_5, a_6, a_7 + 2\delta)$ , where  $\delta = a_0 + 2a_1 + 3a_2 + 4a_3 + 3a_4 + 2a_5 + a_6 + 2a_7$ . Let  $\bar{u} = T(u)$  and  $\underline{v} = T^{-1}(v)$ . Then we have the system of functional equations:

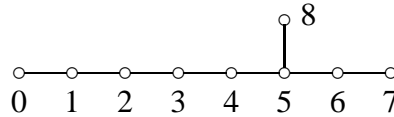
$$\frac{(\underline{v} - u + a_7 - \delta)(v - u + a_7)}{(\underline{v} - u - a_7 + \delta)(v - u - a_7)} \tag{2.2a}$$

$$\begin{aligned} &= \frac{(u + a_3)(u + a_3 + 2a_4)(u + a_3 + 2a_4 + 2a_5)(u + a_3 + 2a_4 + 2a_5 + 2a_6)}{(u - a_3)(u - a_3 - 2a_2)(u - a_3 - 2a_2 - 2a_1)(u - a_3 - 2a_2 - 2a_1 - 2a_0)}, \\ &\frac{(\bar{u} - v - a_7 - \delta)(u - v - a_7)}{(\bar{u} - v + a_7 + \delta)(u - v + a_7)} \tag{2.2b} \\ &= \frac{(v + a_3 + a_7)(v + a_3 + 2a_4 + a_7)(v + a_3 + 2a_4 + 2a_5 + a_7)(v + a_3 + 2a_4 + 2a_5 + 2a_6 + a_7)}{(v - a_3 - a_7)(v - a_3 - 2a_2 - a_7)(v - a_3 - 2a_2 - 2a_1 - a_7)(v - a_3 - 2a_2 - 2a_1 - 2a_0 - a_7)}. \end{aligned}$$

This is called the *difference Painlevé equation of type  $E_7^{(1)}$* .

### 2.3 $E_8^{(1)}$ case ( $\ell_1 = 6, \ell_2 = 3, \ell_3 = 2$ )

The Dynkin diagram of type  $E_8^{(1)} = T_{6,3,2}$  is



Let  $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$  be the root variables of  $E_8^{(1)}$ . The interpretation from the original variables  $a_0, a_i^m$  (in Section 1) is given by

$$a_0 := a_5, \quad a_{\{1,2,3,4,5\}}^1 := a_{\{4,3,2,1,0\}}, \quad a_{\{1,2\}}^2 := a_{\{6,7\}}, \quad a_1^3 := a_8.$$

Define the action of simple reflections  $s_i$  on the root variables by  $s_i(a_j) = a_j - a_i C_{ij}$ , where  $C_{ij}$  is the Cartan matrix of  $E_8^{(1)}$ .

First we blow up the two points  $P_1^2, P_1^3$  and blow down (the proper transform) of the line  $\{F_{h-e_1^2-e_1^3} = 0\}$ . This yields the birational map  $\varphi_1 : \mathbb{P}^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  of the form

$$\varphi_1 : [x_1 : x_2 : x_3] \mapsto (u_1, v_1) = \left( \frac{F_{h-e_1^2-e_1^3}(\mathbf{a}; \mathbf{x})}{x_1 - x_2}, \frac{F_{h-e_1^2-e_1^3}(\mathbf{a}; \mathbf{x})}{x_1 - x_3} \right).$$

Secondly we blow up  $P_2^2 : (u_1, v_1) = (-a_6, \infty)$  and  $P_2^3 : (u_1, v_1) = (\infty, -a_8)$ , and blow down the lines  $\{u_1 = -a_6\}$  and  $\{u_1 = \infty\}$ . Then we have the birational map  $\varphi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  written as

$$\varphi_2 : (u_1, v_1) \mapsto (u_2, v_2) = (u_1 + a_6, (u_1 + a_6)(v_1 + a_8)).$$

Thirdly, by blowing up  $(u_2, v_2) = (0, 0)$  and  $P_3^2 : (u_2, v_2) = (-a_7, \infty)$  and contracting  $\{v_2 = 0\}$  and  $\{v_2 = \infty\}$ , we obtain the birational map  $\varphi_3 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  written as

$$\varphi_3 : (u_2, v_2) \mapsto (u_3, v_3) = \left( \frac{v_2(u_2 + a_7)}{u_2}, v_2 \right).$$

Finally we apply the linear transformation

$$\varphi_4 : (u_3, v_3) \mapsto (u, v) = (4u_3 + (a_6 + a_7 - a_8)^2, 4v_3 + (a_6 - a_8)^2).$$

We see that  $L_1 \cup L_2 \cup L_3$  is generically transformed by  $\varphi = \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$  into the curve  $\{(u - v)^2 - 2a_7^2(u + v) + a_7^4 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree  $(2, 2)$  with a cusp at  $(\infty, \infty)$ .

By virtue of Theorem 1.1, we have the following realization of  $W(E_8^{(1)})$  on  $(u, v)$ :

$$s_6(u) = \tilde{u}, \quad s_7(u) = v, \quad s_7(v) = u,$$

where  $\tilde{u} = \tilde{u}(u, v)$  is a rational function determined by

$$\frac{\tilde{u} - (a_7 + a_8)^2}{\tilde{u} - (a_7 - a_8)^2} = \frac{u - (a_6 + a_7 + a_8)^2}{u - (a_6 + a_7 - a_8)^2} \cdot \frac{v - (a_6 - a_8)^2}{v - (a_6 + a_8)^2}.$$

Alternatively, with respect to the variable (cf. [14])

$$\Gamma = \frac{v - u + a_7^2}{2a_7},$$

we can describe simply the action of  $s_6$  as a linear fractional transformation

$$s_6(\Gamma) = \frac{(v + (a_6 - a_8)(a_6 + a_8))\Gamma + 2a_6v}{2a_6\Gamma + v + (a_6 - a_8)(a_6 + a_8)}.$$

Consider an element

$$T = (s_6 s_5 s_4 s_3 s_2 s_1 s_0 s_8 s_5 s_4 s_3 s_2 s_1 s_6 s_5 s_4 s_3 s_2 s_8 s_5 s_4 s_3 s_6 s_5 s_4 s_8 s_5 s_6 s_7)^2 \in W(E_8^{(1)})$$

acting on the root variables as  $T(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = (a_0, a_1, a_2, a_3, a_4, a_5, a_6 + \delta, a_7 - 2\delta, a_8)$ , where  $\delta = a_0 + 2a_1 + 3a_2 + 4a_3 + 5a_4 + 6a_5 + 4a_6 + 2a_7 + 3a_8$ . Let  $\bar{u} = T(u)$  and  $\underline{v} = T^{-1}(v)$ . Then we have the system of functional equations:

$$\frac{(\bar{u} - v - (a_7 - \delta)^2)(u - v - a_7^2) + 4a_7(a_7 - \delta)v}{2a_7(\bar{u} - v - (a_7 - \delta)^2) + 2(a_7 - \delta)(u - v - a_7^2)} = \frac{G_4(\theta; v)}{G_3(\theta; v)}, \quad (2.3a)$$

$$\frac{(u - \underline{v} + (a_7 + \delta)^2)(u - v + a_7^2) + 4a_7(a_7 + \delta)u}{2a_7(u - \underline{v} + (a_7 + \delta)^2) + 2(a_7 + \delta)(u - v + a_7^2)} = \frac{G_4(\theta + a_7 \mathbf{1}; u)}{G_3(\theta + a_7 \mathbf{1}; u)}, \quad (2.3b)$$

called the *difference Painlevé equation of type  $E_8^{(1)}$* . Here we introduce the polynomials

$$G_4(\theta; v) = v^4 + \sigma_2 v^3 + \sigma_4 v^2 + \sigma_6 v + \sigma_8,$$

$$G_3(\theta; v) = \sigma_1 v^3 + \sigma_3 v^2 + \sigma_5 v + \sigma_7,$$

where  $\sigma_i$  is the  $i$ -th elementary symmetric function of the eight variables  $\theta = (\theta_1, \dots, \theta_8)$  with  $\theta_1 = a_6 - a_8$ ,  $\theta_2 = s_8(\theta_1) (= a_6 + a_8)$ ,  $\theta_3 = s_5(\theta_2)$ ,  $\theta_4 = s_4(\theta_3)$ ,  $\theta_5 = s_3(\theta_4)$ ,  $\theta_6 = s_2(\theta_5)$ ,  $\theta_7 = s_1(\theta_6)$  and  $\theta_8 = s_0(\theta_7)$ , and the symbol  $\mathbf{1}$  denotes  $(1, \dots, 1)$ .

*Remark 2.1.* For reference, we mention some known results about the difference Painlevé equations of type  $E_r^{(1)}$  ( $r = 6, 7, 8$ ); these equations were originally discovered by Grammaticos-Ohta-Ramani [15] (see also good review articles [6, 18] with a list of discrete Painlevé equations). The associated Lax formalisms (or linear problems) were established by Arinkin-Borodin [1] for  $E_6^{(1)}$  and by Boalch [3] for  $E_7^{(1)}$  and  $E_8^{(1)}$ , respectively. It was recently reported by Kajiwara [8] that for all  $E_r^{(1)}$  cases they admit special solutions in terms of the hypergeometric functions in Askey's scheme as well as the  $q$ -difference ones [10].

It often occurs in discrete/continuous Painlevé equations that an interesting class of solutions (written by means of Schur function and its variations) is generated by the Weyl group action from a fixed point of diagram automorphisms. In connection with integrable systems like KP hierarchy, we will study elsewhere such solutions based on the framework of tau-functions (cf. [13, 19]).

*Acknowledgements.* The author wishes to thank Kenji Kajiwara, Tetsu Masuda, Mikio Murata and Yasuhiko Yamada for fruitful discussions. Research of the author is supported in part by JSPS Grant 19840039.

## References

- [1] Arinkin, D., Borodin, A.: Moduli spaces of  $d$ -connections and difference Painlevé equations. *Duke Math. J.* **134**, 515–556 (2006)
- [2] Barth, W., Hulek, K., Peters, C., Van de Ven, A.: Compact complex surfaces. 2nd edn. Berlin: Springer, 2004
- [3] Boalch, P.: Quivers and difference Painlevé equations. arXiv:0706.2634 (preprint)

- [4] Coble, A. B.: Algebraic geometry and theta functions. Amer. Math. Soc., Providence, RI, 1929
- [5] Dolgachev, I., Ortland, D.: Point sets in projective spaces and theta functions. *Astérisque* **165** (1988)
- [6] Grammaticos, B., Ramani, A.: Discrete Painlevé equations: a review. *Lecture Notes in Phys.*, vol. 644, Berlin: Springer, 2004, pp. 245–321
- [7] Kac, V. G.: Infinite dimensional Lie algebras. 3rd edn. Cambridge: Cambridge University Press, 1990
- [8] Kajiwara, K.: Hypergeometric solutions to the discrete Painlevé equations of additive E type. Reports of RIAM Symposium No.19 (2008) (Japanese), in press
- [9] Kajiwara, K., Masuda, T., Noumi, M., Ohta, Y., Yamada, Y.:  ${}_{10}E_9$  solution to the elliptic Painlevé equation. *J. Phys. A: Math. Gen.* **36**, L263–L272 (2003)
- [10] Kajiwara, K., Masuda, T., Noumi, M., Ohta, Y., Yamada, Y.: Hypergeometric solutions to the  $q$ -Painlevé equations. *Int. Math. Res. Not.* **2004**, no. 47, 2497–2521 (2004)
- [11] Kajiwara, K., Masuda, T., Noumi, M., Ohta, Y., Yamada, Y.: Point configurations, Cremona transformations and the elliptic difference Painlevé equation. *Sémin. Congr.* **14**, 169–198 (2006)
- [12] Looijenga, E.: Rational surfaces with an anticanonical cycle. *Ann. of Math. (2)* **114**, 267–322 (1981)
- [13] Masuda, T., Tsuda, T.: On special solutions to the  $q$ -Painlevé equation of type  $E_7^{(1)}$ . Lecture at Kobe workshop on integrable systems and Painlevé systems (2005)
- [14] Murata, M., Sakai, H., Yoneda, J.: Riccati solutions of discrete Painlevé equations with Weyl group symmetry of type  $E_8^{(1)}$ . *J. Math. Phys.* **44**, 1396–1414 (2003)
- [15] Ohta, Y., Ramani, A., Grammaticos, B.: An affine Weyl group approach to the eight-parameter discrete Painlevé equation. *J. Phys. A: Math. Gen.* **34**, 10523–10532 (2001)
- [16] Okamoto, K.: Sur les feuilletages associés aux équation du second ordre à points critiques fixes de P. Painlevé. *Japan J. Math.* **5**, 1–79 (1979) (French)
- [17] Sakai, H.: Rational surfaces associated with affine root systems and geometry of the Painlevé equations. *Comm. Math. Phys.* **220**, 165–229 (2001)
- [18] Sakai, H.: Problem: discrete Painlevé equations and their Lax forms. *RIMS Kokyuroku Bessatsu* **B2**, 195–208 (2007)
- [19] Tsuda, T.: Universal character and  $q$ -difference Painlevé equations with affine Weyl groups. *UTMS* 2005-21 (preprint)
- [20] Tsuda, T.: Tropical Weyl group action via point configurations and  $\tau$ -functions of the  $q$ -Painlevé equations. *Lett. Math. Phys.* **77**, 21–30 (2006)
- [21] Tsuda, T., Takenawa, T.: Tropical representation of Weyl groups associated with certain rational varieties. *arXiv:math/0607661* (preprint)